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LETTER TO THE EDITOR

Three-dimensional field theory with topological and nontopological mass: Hamiltonian and Lagrangian analysis

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Abstract. A three-dimensional (Abelian) gauged massive Thirring model is bosonized in the large fermion mass limit. A further integration of the gauge field results in a nonlocal theory. A truncated version of that is the Maxwell–Chern–Simons (MCS) theory with a conventional mass term or MCS–Proca theory. This theory is completely solved in the Hamiltonian and Lagrangian formalism, with the spectra of the modes determined. Since the vector field constituting the model is identified (via bosonization) with the fermion current, the charge current algebra, including the Schwinger term, is also computed in the MCS–Proca model.

1. Introduction

It has been appreciated for quite some time that gauge symmetry in (2 + 1)-dimensions is subtle, mainly due to the Chern–Simons term [1]. Self-dual theory with the (nontopological) mass term is gauge invariant, being dual to the Maxwell–Chern–Simons (MCS) gauge theory. The Chern–Simons term is referred to as the topological mass term. A master Lagrangian has been constructed [2] which can generate both the above-mentioned models.

Including a nontopological mass term in the MCS model leads to the so-called MCS–Proca (MCSP) model [3]. A Lagrangian analysis was given in [3] where the spectra of two massive modes were provided. In this paper, a detailed Hamiltonian constraint analysis [4] is provided for the first time. It is shown that an involved analysis leads to identical spectra and equations of motion obtained via the Lagrangian method. This is one of our main results. But there are additional benefits of Dirac analysis, which we elaborate below.

Let us now put the MCSP model, studied here, in its proper perspective. Our motivation in the above model is that it has been derived from a three-dimensional $U(1)$ gauged massive Thirring model [5] via bosonization of the fermion modes (in the large fermion mass limit) [1,6]. The bosonic theory is a master Lagrangian, comprised of the $U(1)$ gauge field A_μ, and an auxiliary field B_μ, introduced to linearize the Thirring self-interaction term. Integrating over B_μ leads to a generalized MCS model, which under certain approximations sheds light on the self-interaction effects on the inter-'quark' potential [7]. On the other hand, integration of the gauge field A_μ (in the Lorentz gauge) yields a generalization of the MCSP model in B_μ. A truncated version of it is the MCSP model in question. The added bonus of this scheme is that *B_μ reflects the behaviour of the fermion current $J_\mu = \bar{\psi}\gamma_\mu\psi$ since $J_\mu \equiv B_\mu/g$, g denoting the Thirring coupling*. Indeed, we have correctly reproduced the *current conservation* and *current*

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algebra including the Schwinger term. More complicated fermionic composite objects can also be studied. That is the other important result of this letter.

The letter is organized as follows: first we briefly give the bosonization results of the gauged Thirring model. Then we deal with the Lagrangian formulation, in a way similar to [3]. The particle spectra is obtained. The main body of our work follows, consisting of the full Hamiltonian analysis and current algebra results. The letter ends with a brief conclusion.

2. Bosonization of the gauged Thirring model

The $U(1)$ gauged Thirring model Lagrangian is

$$\mathcal{L}_F = \bar{\psi}\mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}eA_\mu)\psi - m\bar{\psi}\psi + \frac{g}{2}|\bar{\psi}\gamma^\mu\psi|^2 - \frac{pe^2}{4}|A_{\mu\nu}|^2 + \frac{qe^2}{2}\epsilon_{\mu\nu\lambda}A^\mu A^{\nu\lambda}. \tag{1}$$

Here $A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and conventionally one takes $p = 1/e^2$, $q = \mu/(2e^2)$. The reason we have considered them arbitrary will become clear as we proceed. The above model is linearized via the auxiliary field B_μ as

$$\mathcal{L}_F = \bar{\psi}\mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}eA_\mu - \mathrm{i}B_\mu)\psi - \frac{1}{2g}|B_\mu|^2 - m\bar{\psi}\psi - \frac{pe^2}{4}|A_{\mu\nu}|^2 + \frac{qe^2}{2}\epsilon_{\mu\nu\lambda}A^\mu A^{\nu\lambda}. \tag{2}$$

One-loop bosonization in the large fermion mass (as well as small momentum, i.e. lower number of derivatives) limit yields the bosonic Lagrangian (to $\mathrm{O}(\partial/m)$),

$$\mathcal{L}_B = -\frac{a}{4}C_{\mu\nu}C^{\mu\nu} + \frac{\alpha}{2}\epsilon_{\mu\nu\lambda}C^\mu C^{\nu\lambda} - \frac{1}{2g}B_\mu B^\mu - \frac{pe^2}{4}A_{\mu\nu}A^{\mu\nu} + \frac{qe^2}{2}\epsilon_{\mu\nu\lambda}A^\mu A^{\nu\lambda} \tag{3}$$

where $C_\mu = B_\mu + eA_\mu$, $\alpha = -1/(8\pi)$ and $a = -1/(6\pi m)$. The $U(1)$ gauge invariance present in (1) is clearly visible as regards the A_μ field. The A_μ (gauge) and B_μ ('matter') field equations are

$$a\partial_\mu C^{\mu\alpha} + \frac{\alpha}{2}\epsilon^{\alpha\mu\nu}C_{\mu\nu} + ep\partial_\mu A^{\mu\alpha} + \frac{eq}{2}\epsilon^{\alpha\mu\nu}A_{\mu\nu} = 0 \tag{4}$$

$$a\partial_\mu C^{\mu\alpha} + \frac{\alpha}{2}\epsilon^{\alpha\mu\nu}C_{\mu\nu} - \frac{1}{g}B^\alpha = 0. \tag{5}$$

The above two equations are combined to give

$$\frac{1}{g}B^\alpha + ep\partial_\mu A^{\mu\alpha} + \frac{eq}{2}\epsilon^{\alpha\mu\nu}A_{\mu\nu} = 0. \tag{6}$$

Note that without the gauge field kinetic terms in the parent fermion model, we would have obtained simply $B_\mu = 0$.

The Lagrangian in (3) is our *master* Lagrangian [5]. Upon selective integration of the interacting fields in turn, different equivalent (dual) theories are reproduced which are apparently distinct. In this way, it is possible to connect different well known theories. The duality between them appears in the form of a particle spectrum, symmetry, Green's function etc. The next task is to integrate out the gauge field.

3. Particle spectrum: Lagrangian framework

Modulo total derivative terms, A_μ integration in the Lorentz gauge gives [5]

$$\mathcal{L}_B(B_\mu) = B_\mu\frac{\frac{ap(a+p)}{8}\partial^2 + \frac{1}{2}(p\alpha^2 + q^2 a)}{(\frac{p+a}{2})^2\partial^2 + (q+\alpha)^2}(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\mu)B_\nu$$
$$+\frac{B_\mu}{2}\frac{\frac{(p^2\alpha + qa^2)}{4}\partial^2 + q\alpha(\alpha+q)}{(\frac{p+a}{2})^2\partial^2 + (q+\alpha)^2}\epsilon^{\mu\nu\lambda}B_{\nu\lambda} - \frac{1}{2g}B_\mu B^\mu. \tag{7}$$

The equation of motion for B_μ is

$$\frac{\frac{ap(a+p)}{8}\partial^2+\frac{1}{2}(p\alpha^2+q^2a)}{(\frac{p+a}{2})^2\partial^2+(q+\alpha)^2}(g^{\mu\nu}\partial^2-\partial^\mu\partial^\mu)B_\nu+\frac{\frac{(p^2\alpha+qa^2)}{4}\partial^2+q\alpha(\alpha+q)}{(\frac{p+a}{2})^2\partial^2+(q+\alpha)^2}\frac{\epsilon^{\mu\nu\lambda}B_{\nu\lambda}}{2}-\frac{1}{2g}B^\mu. \tag{8}$$

Clearly B_μ obeys the current conservation ($\partial_\mu B^\mu=0$), as is required of the fermion current. Defining the dual of B_μ as

$$^*B_\mu=\tfrac{1}{2}\epsilon_{\mu\nu\lambda}B^{\nu\lambda}$$

we obtain two equations and the Lagrangian in (3) in compact notation as,

$$\begin{aligned}&\mathcal{A}(^*B^\alpha)-\mathcal{D}\partial^2B^\alpha=0 \qquad \mathcal{A}(B^\alpha)+\mathcal{D}(^*B^\alpha)=0\\ &\mathcal{L}_B(B_\mu)=B_\mu\mathcal{A}B^\mu+B_\mu\mathcal{D}\epsilon^{\mu\nu\lambda}\partial_\nu B_\lambda.\end{aligned} \tag{9}$$

The nonlocal operators are

$$\mathcal{A}\equiv\frac{\frac{ap(a+p)}{8}\partial^2+\frac{1}{2}(p\alpha^2+q^2a)}{(\frac{p+a}{2})^2\partial^2+(q+\alpha)^2}\partial^2-\frac{1}{2g} \tag{10}$$

$$\mathcal{D}\equiv\frac{\frac{(p^2\alpha+qa^2)}{4}\partial^2+q\alpha(\alpha+q)}{(\frac{p+a}{2})^2\partial^2+(q+\alpha)^2}. \tag{11}$$

Combining the above equations in (9), we get

$$(\mathcal{A}^2+\mathcal{D}^2\partial^2)B^\alpha=0. \tag{12}$$

Unfortunately, the complicated nature of the operators prohibits further study of the field equation. Let us now consider the approximations we mentioned before.

Keeping within the approximations involved in the bosonization scheme itself we drop $\mathrm{O}(a^2)$ terms. However, with a nonvanishing p (that is, in the presence of the Maxwell term in (1)), the nonlocal nature of the effective theory persists. In the present case we avoid this problem by putting $p=0$ and keeping only the Chern–Simons term in (1). It is to be noted that the Maxwell term induced via bosonization, i.e. the $|C_{\mu\nu}|^2$ in (3), remains. The operators now become

$$\mathcal{A}\approx\left(\frac{q^2a}{2(q+\alpha)^2}\partial^2-\frac{1}{2g}\right) \qquad \mathcal{D}\approx\frac{q\alpha}{q+\alpha}. \tag{13}$$

Hence the B_μ equation reduces to

$$\left[\left(\frac{q^2a}{2(q+\alpha)^2}\partial^2-\frac{1}{2g}\right)^2+\left(\frac{q\alpha}{q+\alpha}\right)^2\partial^2\right]B^\alpha=0. \tag{14}$$

The above equation is 'factorized' in the following form [3]:

$$(\partial^2+M_+^2)(\partial^2+M_-^2)B^\alpha=0. \tag{15}$$

The two values of the effective mass parameter are

$$M_\pm^2=\frac{2(q+\alpha)^2}{q^2a}\left[\frac{\alpha^2}{a}-\frac{1}{2g}\pm\alpha\left(\frac{\alpha^2}{a^2}-\frac{1}{ag}\right)^{\frac{1}{2}}\right]. \tag{16}$$

Substituting the local expressions for $\mathcal{A}$ and $\mathcal{D}$, we arrive at the MCSP model by neglecting $\mathrm{O}(a^2)$ terms, but in the above analysis we have not dropped $\mathrm{O}(a^2)$ terms. There is no contradiction here since now we are studying the MCSP model as such, forgetting how it was originated in the first place. However, if we persist with $a^2\approx0$ in (14), we end up with a single massive mode,

$$(\partial^2+M^2)B^\alpha=0 \qquad M^2\approx\frac{(q+\alpha)^2}{16q^2\alpha^2g^2}\left(1+\frac{a}{8\alpha^2g}\right).$$

This concludes the Lagrangian analysis of the MCSP model.

4. Particle spectrum: Hamiltonian framework

We start with the MCSP Lagrangian, using (13),

$$\mathcal{L} = PB_{\mu\nu}B^{\mu\nu} + Q\epsilon_{\mu\nu\lambda}B^{\mu}\partial^{\nu}B^{\lambda} + RB_{\mu}B^{\mu} \tag{17}$$

where

$$P = \frac{-q^2 a}{4(q+\alpha)^2} \qquad Q = \frac{q\alpha}{q+\alpha} \qquad R = -\frac{1}{2g}.$$

The conjugate momenta and the canonical Hamiltonian are defined in the standard way,

$$\Pi_{\mu} = \frac{\partial L}{\partial \dot{B}^{\mu}} \qquad \mathcal{H} = \Pi^{\mu}\dot{B}_{\mu} - \mathcal{L}. \tag{18}$$

Explicit expressions for the above are

$$\Pi_0 = 0 \qquad \Pi_i = -4P(\partial_i B_0 - \dot{B}_i) - Q\epsilon_{ij}B_j \tag{19}$$

$$\mathcal{H} = -\frac{1}{8P}(\Pi_i + Q\epsilon_{ij}B_j)^2 + B_0(\partial_i\Pi_i - Q\epsilon_{ij}\partial_i B_j) - PB_{ij}B_{ij} - RB_{\mu}B^{\mu}. \tag{20}$$

We now perform the constraint analysis by obtaining the constraints and subsequently computing the Dirac brackets. Our aim is to obtain the equations of motion of the modes and reproduce the spectra obtained in (16). The primary constraint is

$$\Psi_1(x) \equiv \Pi_0(x) \approx 0 \tag{21}$$

and time persistence generates the secondary constraint

$$\Psi_2(x) \equiv \dot{\Psi}_1(x) = \left[\Psi_1(x), \int \mathrm{d}^2 y\, \mathcal{H}(y)\right] = \partial_i\Pi_i(x) - 2RB_0(x) - Q\epsilon_{ij}\partial_i B_j(x) \approx 0. \tag{22}$$

These brackets are obtained by using the fundamental Poisson brackets

$$[\Pi^{\mu}(x), B_{\nu}(y)] = g^{\mu}_{\nu}\delta(x-y).$$

The constraints constitute a second-class pair with the nontrivial algebra

$$[\Psi_1, \Psi_1] = [\Psi_2, \Psi_2] = 0 \qquad [\Psi_1(x), \Psi_2(y)] = 2R\delta(x-y). \tag{23}$$

The inverse of the constraint matrix Ψ_{ij}, defined by $\int \mathrm{d}^2 y\, C_{ij}(x,y)\Psi_{jk}(y,z) = g_{ik}\delta(x-z)$ has the nonzero element

$$C_{12}(x,y) = -\frac{1}{2R}\delta(x-y).$$

This generates the nontrivial Dirac brackets

$$[B_0(x), B_i(y)] = \frac{1}{2R}\partial_i\delta(x-y) \qquad [B_0(x), \Pi_i(y)] = -\frac{Q}{2R}\epsilon_{ij}\partial_j\delta(x-y). \tag{24}$$

The rest of the brackets are not altered. From now on we will always use these Dirac brackets. *The $B_0 - B_i$ bracket recovers the correct Schwinger term in the J_0–fermion current algebra,*

$$[J_0(x), J_0(y)] = [J_i(x), J_j(y)] = 0 \qquad [J_0(x), J_i(y)] = -\frac{1}{g}\partial_i\delta(x-y). \tag{25}$$

After strong implementation of the constraints $\mathcal{H}$ in (20) simplifies to

$$\mathcal{H} = -\frac{1}{8P}(\Pi_i + Q\epsilon_{ij}B_j)^2 - PB_{ij}^2 + R(B_0^2 + B_i^2). \tag{26}$$

The time derivative of B_0 reproduces the current conservation,

$$\dot{B}_0(x) = [B_0(x), H] = \partial_i B_i(x). \tag{27}$$

The above current algebra and conservation equation are two of our main results. Note that the Dirac brackets are crucial in recovering them. We now rederive the particle spectrum.

First we compute the following time derivatives:

$$\dot{B}_i = \frac{1}{4P}(\Pi_i + Q\epsilon_{ij}B_j) + \partial_i B_0 \tag{28}$$

$$\dot{\Pi}_i = -\frac{Q}{4P}(\epsilon_{ji}\Pi_j + QB_i) - 4P\partial_j B_{ij} - Q\epsilon_{ij}\partial_j B_0 + 2RB_i. \tag{29}$$

We take time derivatives of the above equations,

$$\ddot{B}_i = \frac{1}{4P}(\dot{\Pi}_i + Q\epsilon_{ij}\dot{B}_j) + \partial_i \dot{B}_0$$

$$\ddot{\Pi}_i = -\frac{Q}{4P}(\epsilon_{ji}\dot{\Pi}_j + Q\dot{B}_i) - 4P\partial_j \dot{B}_{ij} - Q\epsilon_{ij}\partial_j \dot{B}_0 + 2R\dot{B}_i.$$

A long algebra yields the following set of equations:

$$\left[\partial^2 - \frac{1}{2P}\left(R - \frac{Q^2}{4P}\right)\right]B_i = \frac{Q}{8P^2}\epsilon_{ij}\Pi_j \tag{30}$$

$$\left[\partial^2 - \frac{1}{2P}\left(R - \frac{Q^2}{4P}\right)\right]\Pi_i = -2Q\epsilon_{ij}\partial_j(\partial_k B_k) + \frac{Q}{2P}\left(2R - \frac{Q^2}{4P}\right)\epsilon_{ij}B_j$$
$$+2Q\epsilon_{ik}\partial_j\partial_j B_k - 2Q\partial_i(\epsilon_{jk}\partial_j B_k). \tag{31}$$

The constraints have been used strongly. The same operator arising in the left-hand side of both the above equations is used once again and we get

$$\left[\partial^2 - \frac{1}{2P}\left(R - \frac{Q^2}{4P}\right)\right]^2 B_i = -\frac{Q^2}{16P^3}\left(2R - \frac{Q^2}{4P}\right)B_i. \tag{32}$$

The identical equation appears for Π_i as well. This chain of derivatives diagonalizes the equations of motion. Factorizing (32), we obtain the *identical* set of expressions for $M_\pm$ given in (16). This concludes the Hamiltonian analysis.

Substituting the known expressions we get the explicit forms of the masses,

$$M_\pm^2 = \frac{24\pi e^4 m}{\mu^2}\left(\frac{-1}{8\pi} + \frac{\mu}{2e^2}\right)^2\left[\frac{3m}{16\pi} + \frac{1}{g} \pm \frac{1}{2\pi}\left(\frac{9m^2}{16} + \frac{6\pi m}{g}\right)^{\frac{1}{2}}\right]. \tag{33}$$

Assuming that the relative strengths of $1/m$ and g (the Thirring coupling) is such that for large m, mg (a dimensionless quantity) is also large, we expand the square root and keep terms up to $\mathrm{O}(1/(mg))$, and we get

$$\begin{aligned} M_+^2 &\approx \frac{48\pi e^4 m}{\mu^2}\left(\frac{-1}{8\pi} + \frac{\mu}{2e^2}\right)^2\left(\frac{3m}{4\pi} + \frac{1}{g}\right) \\ M_-^2 &\approx \frac{16\pi^2 e^4}{\mu^2 g^2}\left(\frac{-1}{8\pi} + \frac{\mu}{2e^2}\right)^2. \end{aligned} \tag{34}$$

Interestingly, $M_+ \gg M_-$ since M_- is *independent* of the fermion mass m, the large parameter. Since spin of the particles is determined by the sign of the mass [1, 3], the small value of M_- can lead to a spinless particle.

5. Conclusions

The Hamiltonian and Lagrangian of the MCSP model has been performed, with the full spectra of modes revealed. The interest in the model lies in the fact that the model has been derived from the bosonized version of a $U(1)$ gauged massive Thirring model. Since the bosonic vector field and the fermion current are identified, the bosonized model, and in turn the MCSP model, yields properties of its fermion counterpart. As a nontrivial application of the above, we have computed correctly the fermion current algebra, with the Schwinger term, staying in the MCSP model framework. The behaviour of other fermionic composite objects, constructed from fermion currents, can also be studied in the MCSP model, where the quantum effects enter via the process of bosonization.

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